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An Exponential Total Variation Functional to Replace the Total Variation Norm

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ABSTRACT

In iterative image restoration, the total variation (TV) norm is the most common regulator to control noise and enforce sharp edges in the images. This paper proposes an exponential version of the TV norm. This exponential version of the TV norm behaves similar to the L_p norm for $0 < p < 1$ and it promotes sharp image edges better than the TV norm. This exponential TV is easy to implement. We use computer simulations to compare the effectiveness of preserving sharp edges by the TV regulator and by its exponential version. The image restoration task is image denoising. The exponential regulator is demonstrated by computer simulations as more effective in obtaining piecewise-constant images than the popular TV regulator.

Keywords

Image restoration, Total variation prior, Piecewise constant, Image denoising, L_p norm.

Introduction

Iterative image processing and iterative image reconstruction usually require regularization due to the ill condition of the inverse problems. It can be readily verified that the total variation (TV) norm is not as effective in enforcing the piecewise-constant nature of the image as some other norms, such as the L_p norm with $0 < p < 1$.

For example, consider two monotonically increasing one-dimensional functions $f_1(x)$ and $f_2(x)$ on $[a,b]$. We assume that these two functions have the same values at the end points, that is, $f_2(a)=f_1(a)$ and $f_2(b)=f_1(b)$. Here $f_1(x)$ is a piecewise-constant function, while $f_2(x)$ is a straight line or a smooth curve. The total variation for both functions is the same, with the value of $f_2(b)-f_2(a)=f_1(b)-f_1(a)$. Therefore, the TV norm is not effective to select $f_1(x)$ over $f_2(x)$ if both functions are valid according to certain data fidelity conditions. This shortcoming is due to the linearity of the L_1 norm.

The L_p norm with $0 < p < 1$, on the other hand, is nonlinear in nature. When $p \neq 1$, $|\nabla f|^p$ is nonlinear with respect to ∇f. The L_p^p values for two different discrete functions $y=f_1'(x)$ and $y=f_2'(x)$ are different:

$$L_p^p(f_1') = [f_1(x_n) - f_1(x_{n-1})]^p + [f_1(x_{n-1}) - f_1(x_{n-2})]^p + \cdots + [f_1(x_2) - f_1(x_1)]^p, \quad (1)$$

$$L_p^p(f_2') = [f_2(x_n) - f_2(x_{n-1})]^p + [f_2(x_{n-1}) - f_2(x_{n-2})]^p + \cdots + [f_2(x_2) - f_2(x_1)]^p. \quad (2)$$

For different functions, we have

$$L_p^p(f_1') \neq L_p^p(f_2'). \quad (3)$$

What is the ideal value of p? If our task is to encourage the piecewise nature of the curve and penalize the smooth transitions, the penalty function should encourage zero-jump transitions and pay attention to small non-zero transitions (e.g., what is preferred and what is not preferred). When the transitions are large enough, they should be treated almost the same. Figure 1 shows some functions of $y=|x|^p$. Small positive values of p satisfy our needs.

The L_p norm is not easy to implement in an iterative algorithm, because its gradient has a singularity with the value of infinity at the origin when $p<1$. In fact, when $p<1$ and

$$y=|x|_p, \quad (4)$$

then,

$$\frac{dy}{dx} = \frac{p}{|x|^{1-p}}. \quad (5)$$

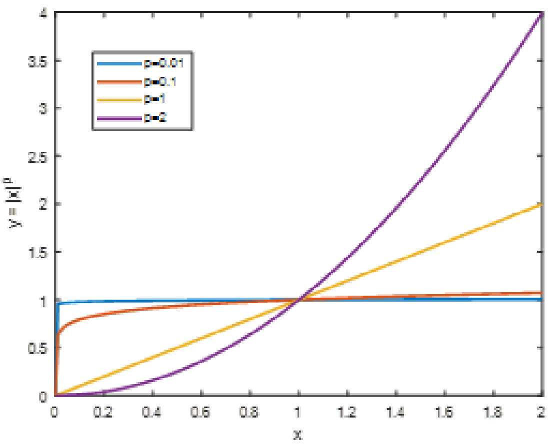

Figure 1: Functions of $y=|x|^p$ over [0, 2] with some values of p.

To avoid the singularity at $x=0$ in (5), this paper suggests an alternative functional to replace the L_p norm. This alternative functional is a modification of the well-known TV norm [1-3].

Methods

For an image X, the isotropic TV norm is defined as

$$TV(X) = \sum_{i,j} T_{i,j}\,, \tag{6}$$

with

$$T_{i,j} = \sqrt{\left(x_{i+1,j} - x_{i,j}\right)^2 + \left(x_{i,j+1} - x_{i,j}\right)^2}\,, \tag{7}$$

where $x_{i,j}$ is a pixel in image X. Eq. (7) defines a *local* L_2 norm of the image gradient. Eq. (6) is the L_1 norm of the elements $\{T_{i,j}\}$. The proposed exponential TV functional is defined as

$$expTV(X) = \sum_{i,j} \frac{1 - \exp(-aT_{i,j})}{1 - \exp(-a)}. \tag{8}$$

The proposed function in (8) contains a user-defined parameter a. A scalar version of each term in (8) is

$$f(x) = \frac{1 - \exp(-a|x|)}{1 - \exp(-a)} \tag{9}$$

and is plotted with four different values of parameter a in Figure 2. The function $f(x)$ in (9) has the following properties when $0<x<1$:

$$f(0) = 0, \tag{10}$$

$$f(1) = 1, \tag{11}$$

$$f'(x) > 0, \tag{12}$$

$$f''(x) < 0, \tag{13}$$

which are the same properties as the function

$$g(x) = |x|^p. \tag{14}$$

The function $g(x)$ defined in (14) is the definition of the L_p norm for the scalar case. The function $g(x)$ in (14) has a parameter p and is plotted in Figure 1 with four different values of p.

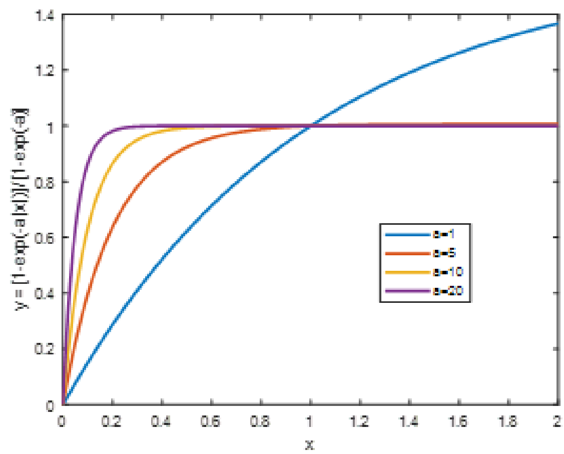

Figure 2: Some curves of the proposed function with different values of parameter a.

By comparing (Figure 1 and Figure 2), we notice that the curves in Figure 1 have slopes of almost infinity, which is not user-friendly in an iterative algorithm. Also, a small p in (14) corresponds to a large a in (8). We can define unit circles as

$$f(x) + f(y) = 1, \tag{15}$$

and

$$g(x) + g(y) = 1, \tag{16}$$

respectively. The unit circles for f and g are compared in Figure 3 LEFT and RIGHT. The general shapes of these unit circles are similar.

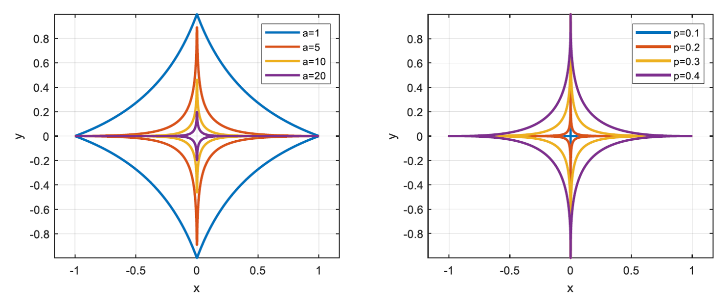

Figure 3: LEFT: Some curves of the proposed function with different values of parameter a. **RIGHT:** Some curves of the Lp norm with different values of parameter p.

Similar to the situation of the L_0 'norm,' $f(x)$ defined in (8) is not a norm either, due to the fact that the absolute homogeneity property

$$f(kx) \neq |k| \cdot f(x). \tag{17}$$

does not hold, that is,

In the next section, we will show the different results of the one-dimensional (1D) TV norm and the 1D expTV function when both are applied to some curves. We will also show their results when they are used as a regulator in a two-dimensional (2D) image denoising task.

Results

The one-dimensional (1D) versions of (6) and (8) are given as

$$TV(X) = \sum_i |x_{i+1} - x_i|, \tag{18}$$

and

$$expTV(X) = \sum_i \frac{1 - \exp(-a|x_{i+1} - x_i|)}{1 - \exp(-a)}. \tag{19}$$

Here we consider two curves in Figure 4. One is a smooth sine-wave pulse and the other is a piecewise-constant rectangular pulse. According to (18) the TV values for both curves in Figure 4 are the same. This implies that the TV norm cannot be used to distinguish these two curves and the TV constraint is not effective in enforcing the piecewise-constant nature.

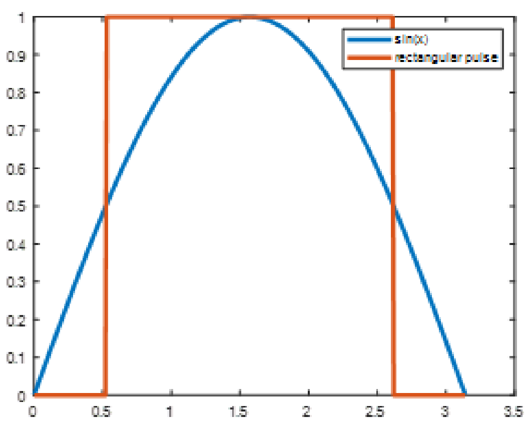

Figure 4: A sine-wave and a rectangular pulse.

On the other hand, if we apply the newly proposed expTV (19) to these two curves, the piecewise-constant curve has a much smaller value than the smooth sine wave as shown in Table 1. An observation is that for a larger parameter a, the difference between the rectangular pulse expTV value and the sine-wave pulse expTV value is greater.

	Sine wave	Rectangular pulse
TV	2	2
expTV with a = 1	3.1491	2
expTV with a = 5	9.8651	2
expTV with a = 10	19.2221	2
expTV with a = 20	36.9997	2

Table 1: ExpTV values for two 1D curves shown in Figure 4.

Next, we consider an image denoising task with an objective function

$$H(X) = \left\|A(X - \hat{X})\right\|_{L_2}^2 + \lambda \cdot \text{regulator}(X), \tag{20}$$

where $\hat{X}$ is the input noisy image, X is the denoised image, A is a 3×3 lowpass filter kernel, λ is a user specified parameter to balance the data fidelity term and the Bayesian term, and regulator(X) is either $TV(X)$ or $expTV(X)$ as defined in (6) and (8), respectively.

An iterative gradient descent algorithm was used to minimize the objective function (20). The 256×256 Shepp-Lagon phantom was used in the computer simulation. The parameter λ was chosen as 0.0005, and the number of iterations was set at 2000. The results are shown in Figure 5. The Structural Similarity Index Measure (SSIM) was used to compare the images (see Table 2) [4]. A larger SSIM value indicates a better image. It is observed that the expTV functional is more effective than the TV norm in preserving the piecewise-constant nature of the image.

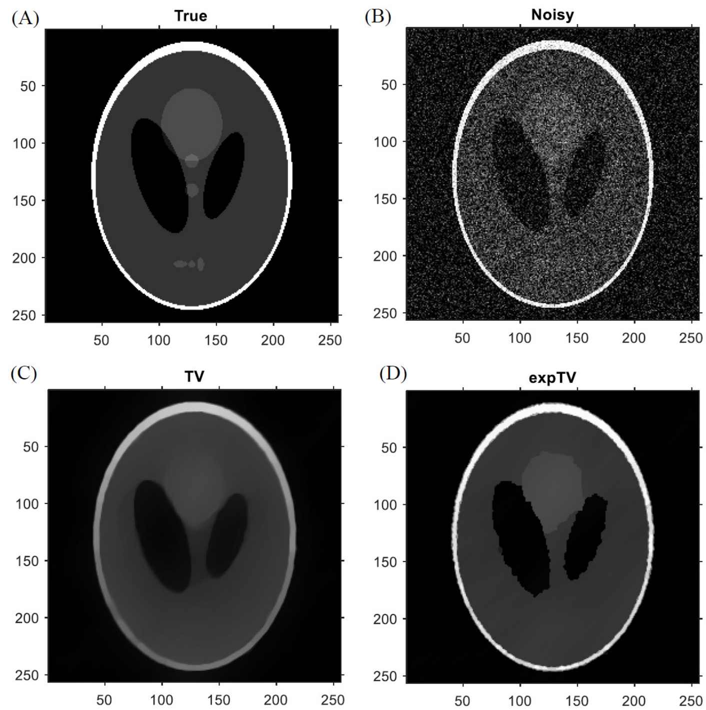

Figure 5: (A) True phantom; (B) Noisy image; (C) TV denoising; (D) expTV denoising.

Ture image	Noisy Image	TV Denoising	expTV Denoising
1.0000	0.1040	0.6546	0.8743

Table 2: SSIM.

Conclusion

The TV norm is unable to distinguish a smooth monotonic function and a piecewise-constant monotonic function. On the other hand, the L_p norm can effectively distinguish a smooth monotonic function and a piecewise-constant monotonic function when $0 \leq p<1$. This is due to the concave down shape the curve $y=x^p$ for $x>0$, which can be easily verified by $dy/dx>0$ and $d^2y/dx^2<0$. The basic requirements for the penalty of the image gradient are: No penalty for zero gradient. Almost the same penalty for the image gradients large enough to be considered as edges.

The L_p norm is difficult to implement in a gradient descent algorithm due to a singularity at the origin for $0 \leq p<1$. Based on the TV norm, this paper suggests an exponential TV functional to replace the L_p norm. Our proposed exponential TV functional behaves in a similar way as the L_p norm for $0<p<1$ and is easy to implement. There is no singularizes in the gradient of the proposed exponential function.

Even though the proposed exponential TV functional is shown to have almost all properties of the L_p norm, exponential TV functional is not a norm. In fact, a regulator does not need to be a norm. The effectiveness of the exponential TV functional is demonstrated with computer simulations of a denoising task.

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References

1. Rudin Leonid I, Osher Stanley, Fatemi Emad. "Nonlinear total variation-based noise removal algorithms." Physica D: Nonlinear Phenomena. 1992; 60: 259-268.
2. Blomgren Peter, Chan Tony F. "Color TV: total variation methods for restoration of vector-valued images." IEEE Trans Image Process. 1998; 7: 304-309.
3. Tony F Chan, Jackie (Jianhong) Shen. Image Processing and Analysis - Variational, PDE, Wavelet, and Stochastic Methods, SIAM. 2005.
4. Wang Zhou, Bovik AC, Sheikh HR, et al. "Image quality assessment: from error visibility to structural similarity." IEEE Trans Image Process. 2004; 13: 600-612.

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